11/18 Lecture Notes

- When is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ not invertible?
  - ↳ Whenever $\frac{1}{ad-bc} = \frac{1}{0}$
- **Determinant** ⇒ Arithmetic condition (number, not a matrix) for determining if a matrix is invertible

geometrically $\det(A)$ = scaling factor for dimension volumes

- When is 1×1 matrix $[a]$ invertible? ⇒ when $a \neq 0$
- **Idea**: The determinant is a function that inputs a square matrix and outputs a number
- **Definition**: • $\det([a]) = a$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

- If A is an $n\times n$ matrix let M_{ij} denote the matrix you get from deleting the ith row & jth column of A

Ex) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow M_{23} = \begin{bmatrix} 1 & 2 & \cancel{3} \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & 8 & \cancel{9} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$

- Let $C_{ij} = (-1)^{i+j}\det(M_{ij}) \Rightarrow C_{23} = (-1)^5 \det\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$

$C_{23} = -1((1)(8) - (2)(7)) = 6$

- **Definition**: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \Rightarrow$ then $\det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

$M_{ij} = M_{11}$

T = even # of reflections across planes
$-$ = odd

Ex)

$\det\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}\right) \Rightarrow \det = \underset{a_{11}}{(1)}\cdot\left(\underset{C_{11}}{(-1)^2\det\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}}\right) + (2)\left((-1)^3\det\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}\right) + (0)\left((-1)^4\det\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}\right)$

$+ 0(-1)^5\det\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\Big) \Rightarrow \det(A) = \det\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}}_{Q_1} + (2)(-1)\det\underbrace{\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}}_{Q_2}$

$\det Q_1 = (1)\left((-1)^2\det\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\right) + 0 + 2\left((-1)^4\det\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) = -1 + 0 + 2(1) = 1$

$\det Q_2 = 2\left((-1)^2\det\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\right) + 0 + 2\left((-1)^4\det\begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}\right) = (2)(1)(-1) + 0 + (2)(1)(0) = -2$

$\det(A) = (1) + (-2)(-2) = 5$

Theorem: Let A be the same matrix as above then (a) det

(a) $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots a_{2n}C_{in}$ (across a fixed row)

(b) $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots a_{nj}C_{nj}$ (down a fixed column)

Ex) $\det\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}\right) \Rightarrow \det(A) = 0 + 0 + 0 + 1\left((-1)^{3+4}\det\left(\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\right)\right)$

$= -\left(0 + 1\left((-1)^5\det\left(\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\right)\right) + 1\left((-1)^6\det\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)\right)$

$= -(-2 + (1\cdot1 - 2\cdot2)) = 5$

Theorem: $\det(I_n) = 1$

Theorem: Let A be an $n\times n$ matrix, then $\det(A) \neq 0$ iff A is invertible